Indian Statistical Institute
M.Math. II Year
Second Semester 2006-07
Mid Semester Examination
Stochastic Processes II

Date: 28-02-07 Max. Score 70 Time: 3 hrs

This is an 'open notes' examination.

1. a) Let $\{X_\lambda,\ \lambda \in \mathbb{R}\}$ be a real valued mean zero, L^2-process with orthogonal increments i.e. if $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ then $E(X_{\lambda_2} - X_{\lambda_1})(X_{\lambda_4} - X_{\lambda_3}) = 0$. Suppose there exists $a > 0$ such that

$$\lim_{\substack{\lambda_1 \uparrow \infty \\ \lambda_2 \downarrow -\infty}} E\, |X_{\lambda_1} - X_{\lambda_2}|^2 = a.$$

Show that there exists a measure μ on $(\mathbb{R}, \mathcal{B})$ and an $X : \mathcal{B} \to L^2$, a measure with orthogonal values (with associated measure μ) such that if $A = [\lambda_1, \lambda_2]$ then $X(A) = X_{\lambda_2} - X_{\lambda_1}$. (10)

b) Let $F : \mathbb{R} \to [0, 1]$ be a continuous probability distribution function. Show that there exists a mean zero Gaussian process $\{X_\lambda, \lambda \in \mathbb{R}\}$ such that $E\, X_{\lambda_1} X_{\lambda_2} = F(\lambda_1)$ if $\lambda_1 < \lambda_2$. (10)

c) Show that the process in b) has orthogonal increments. (5)

2. Let $\{X_t,\ t \in [a, b]\}$ be a mean zero L^2-process with a continuous covariance function $K : [a, b] \times [a, b] \to \mathbb{C}$. Express the following expected values in terms of K and prove your results:

a) $E \left| \int_a^b X_s\, ds \right|^2$ b) $E \left(X_t \overline{\int_a^b X_s\, ds} \right)$. (5 + 5)

3. a) Let T be a measure preserving transformation on $(\Omega, \mathcal{F}, \mu)$. Show that if T is mixing w.r.t. μ then $\mu(\Omega) = 1$. (9)

b) Show that if T is mixing then T is ergodic. (6)

c) Suppose the mixing property holds for all $A, B \in \mathcal{F}_0$ where $\mathcal{F}_0$ is a field. Then show that it holds for all $A, B \in \sigma(\mathcal{F}_0) = \mathcal{F}$. (10)

4. Let ν be the standard Gaussian measure on $(\mathbb{R}, \mathcal{B})$. Show that if $f \in L^2(\nu)$ then $f = \sum_{n=0}^{\infty} C_n H_n$; where $\{H_n\}$ are the Hermite polynomials, C_n are constants and equality holds in $L^2(\nu)$. (10)